No Spontaneous Breakdown of Chiral Symmetry in Nambu-Jona-Lasinio Model?

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We argue that the spontaneous breakdown of symmetry in the chirally symmetric Nambu–Jona-Lasinio model which was supposed to illustrate the origin of the low mass of pions in hadron physics does not occur due to strong fluctuations in the σ - π field space. Although quarks acquire a constituent mass, σ and π turn out to have equal heavy masses of the order of the constituent quark mass.

I. INTRODUCTION

The chirally symmetric Nambu–Jona-Lasinio model [1] was the first theoretical laboratory to illustrate how light pions arise from a spontaneous breakdown of chiral symmetry in hadron physics. The first realistic formulation of the model which included flavored quarks, possessed chiral symmetry $SU(3) \times SU(3)$, and a spectrum of σ, π, ρ, A_1 mesons and their $SU(3)$ partners, was formulated and investigated in 1976 by one of the authors [2], and has been the source of inspiration for many papers in nuclear physics in the past twenty years [3]. By eliminating the Fermi fields in favor of a pair of collective scalar and pseudoscalar fields σ and π, as well as vector and axial vector mesons, a Ginzburg Landau like collective field action was derived. This had been studied in detail earlier as an effective action guaranteeing all low-energy properties of hadronic strong interactions which were known from current algebra and partial conservation of the axial current (PCAC).

In two important respects, however, the model was unsatisfactory. First it was not renormalizable in four dimensions but required a momentum space cutoff Λ to produce finite results. Moreover, to obtain physical quantities of the correct size, the cutoff had to be rather small, below one GeV, thus limiting the reliability of the predictions to very low energies. Second, if the fermions were identified with quarks, the model could not account for their confinement.

The nonrenormalizability was removed in [2] by replacing the four-fermion interaction by the exchange of a massive vector meson. The different attractive meson channels were obtained by a Fierz transformation. The mass of the vector meson took over the role of the cutoff. The energy range of applicability was, however, not increased since the model would still allow for free massive quarks.

The purpose of this note is to point out a much more severe problem with the model which seems to invalidate most conclusions derived from it in the literature: If chiral fluctuations are taken into account in a certain nonperturbative approximation, the spontaneous symmetry breakdown disappears, and the zero-mass pions acquire the same mass as the σ-mesons. The nonperturbative nature of the argument seems to be the reason why the phenomena has been overlooked until now.

Since the Nambu–Jona-Lasinio model is incapable of accounting for confinement, it gave no reason for introducing colored quarks. It is curious to observe that the restoration of symmetry by chiral fluctuations would offer such a reason, albeit with an unphysical number of colors: our conclusion can be avoided only by introducing at least five identical replica of fermions. The existing three colors are therefore insufficient to save the model within our approximation.

The non-perturbative arguments used in this paper are anologous to those applied before in a discussion of the Gross-Neveu model [4] in $2 + \varepsilon$ dimensions [5] where it was shown that this model has two phase transitions, one where quarks become massive and another one where chiral symmetry breaks spontaneously.

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II. NAMBU–JONA-LASINIO MODEL

Let us briefly recall the relevant features of the the Nambu–Jona-Lasinio model for our considerations. The model contains N_f quark fields $\psi(x)$, one for each flavor. Later, when we run into the difficulties in achieving spontaneous symmetry breakdown, we shall call for N_c identical replica of the Fermi fields, which we shall therefore introduce into the model right in the beginning, thus dealing with $N = N_f \times N_c$ degenerate fields of colored quarks. Since the phenomenon to be discussed will be caused by the almost massless modes, we may restrict ourselves to the almost massless up and down quarks. We will comment later on the effect of the other quarks.

The Lagrangian of the model is therefore given by

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m_0\right)\psi + \frac{g_0}{2N_c}\left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}\lambda_a i\gamma_5\psi\right)^2\right] \tag{1}$$

where an implicit summation over $a = 1, 2, 3$ is assumed, if not stated otherwise. This Lagrangian is not $U_A(1)$ invariant and is then an effective description of the strong axial anomaly.

A small diagonal quark mass matrix m_0 breaks slightly the $SU(2)\times SU(2)$ part of the chiral symmetry which lifts the mass of the pion to a small nonzero value. We have omitted the more realistic flavor symmetric, vector gluon exchange which would have given rise, after a Fierz transformation, to additional vector and axial vector interactions, thus concentrating on the relevant scalar and pseudoscalar channels. We also use the original nonrenormalizable interaction corresponding to an infinite vector gluon mass, since the chiral fluctuations to be investigated do not depend on such details. The coupling constant appears with the number of colors in the denominator, N_c, for the academic reason that the model has a finite $N_c \to \infty$ limit at fixed g_0. The 2×2-dimensional matrices $\lambda_a/2,\ a = 1, \ldots, 3$ generate the fundamental representation of flavor $SU(2)$, and are normalized by $\mathrm{tr}(\lambda_a\lambda_b) = 2\delta_{ab}$.

Via a Hubbard-Stratanovich transformation, (1) is equivalent to a theory involving collective scalar and pseudoscalar fields σ and π_a:

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m_0 - \sigma - i\gamma_5\lambda_a\pi_a\right)\psi - \frac{N_c}{2g_0}\left(\sigma^2 + \pi_a^2\right). \tag{2}$$

Defining the propagator in the presence of the meson fields,

$$G \equiv \frac{i}{i\partial\!\!\!/ - m_0 - \sigma - i\gamma_5\lambda_a\pi_a}, \tag{3}$$

and adding quark sources, the Lagrangian can be rewritten as

$$\mathcal{L} \equiv \bar{\psi}iG^{-1}\psi - \frac{N_c}{2g_0}\left(\sigma^2 + \pi_a^2\right) + \bar{\psi}\eta + \bar{\eta}\psi. \tag{4}$$

The generating functional of all Green functions is

$$\mathcal{Z} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \exp\left(i\int d^Dx\mathcal{L}\right). \tag{5}$$

Integrating out the quark fields, the generating functional takes the well-known form

$$\mathcal{Z} = \int \mathcal{D}\sigma\mathcal{D}\pi e^{i\mathcal{A}[\sigma,\pi]}, \tag{6}$$

where

$$\mathcal{A}[\sigma,\pi] \equiv -i\mathrm{Tr}\ln iG^{-1} - \frac{N_c}{2g_0}\int d^Dx\ \left(\sigma^2 + \pi_a^2\right) + i\int d^Dxd^Dy\,\bar{\eta}G\eta \tag{7}$$

denotes the collective field action whereas the symbol $\mathrm{Tr} \equiv \int d^Dx\,\mathrm{tr}$ includes also the functional space-time "index" x. The symbol tr is for trace over color, spin and flavor indices.

We shall also define a reduced trace $\mathrm{Tr}' \equiv N_c^{-1} \int d^D x \, \mathrm{tr}_\gamma \mathrm{tr}_f$, and write the generating functional (6) as

$$\mathcal{Z} = \int \mathcal{D}\sigma\mathcal{D}\pi \exp\left\{ iN_c \left[-i\mathrm{Tr}' \ln iG^{-1} - \frac{1}{2g_0} \int d^D x \, \left(\sigma^2 + \pi_a^2\right) + \frac{i}{N_c} \int d^D x d^D y \, \bar{\eta} G \eta \right] \right\}. \tag{8}$$

By extremizing $\mathcal{A}[\sigma, \pi]$ at zero external currents $\eta, \bar{\eta}$, we obtain the field equation for the collective field (σ, π_a):

$$\mathrm{tr}_\gamma \mathrm{tr}_f \left[G(x,x) \begin{pmatrix} 1 \\ i\lambda_a \gamma_5 \end{pmatrix} \right] = \frac{1}{g_0} \begin{pmatrix} \sigma(x) \\ \pi_a(x) \end{pmatrix}. \tag{9}$$

For constant fields, this equation becomes a *gap equation*. Its solutions will be marked by a superscript "s" for "stationary". From now on, unless explicitely stated, we shall consider the limit $m_0 = 0$. The pseudoscalar solutions π_a^{s} can always be chosen to be vanishing, while the scalar solutions can be $\sigma^{\mathrm{s}} = 0$, or $\sigma^{\mathrm{s}} \equiv \Sigma_0$. In the first solution, the ground state is chirally symmetric, in the second the symmetry is spontaneously broken. This is the state of physical interest whose properties we shall now discuss in more detail.

III. EFFECTIVE POTENTIAL AND GAP EQUATION

In the limit $N_c \to \infty$, the generating functional is given *exactly* by the extremal field configurations which can be parametrized as $(\sigma^{\mathrm{s}}(x), \pi_a^{\mathrm{s}}(x)) = (\Sigma_0(x), 0)$. The system behaves classically, with the effective action per quark

$$\frac{\Gamma(\Sigma_0, \Psi, \bar{\Psi})}{N_c} = -i\mathrm{Tr}' \ln iG_{\Sigma_0}^{-1} - \frac{1}{2g_0} \int d^D x {\Sigma_0}^2 + \frac{1}{N_c} \int d^D x \bar{\Psi}_a i G_{\Sigma_0}^{-1} \Psi_a, \tag{10}$$

where $\Psi = iG_{\Sigma_0}\eta$ is the expectation value $< \psi >$ of the quark field and G_{Σ_0} its propagator

$$G_{\Sigma_0} = \frac{i}{i\partial\!\!\!/ - \Sigma_0} \tag{11}$$

This shows that the solution of the gap equation with $\Sigma_0 \neq 0$ corresponds to massive quarks, which have been generated by the spontaneous symmetry breakdown. The mass of the quarks, to be denoted by M, is equal to Σ_0, and is referred to as the constituent quark mass. In the present approximation of zero bare mass m_0, the contituent quark mass is about equal to 300 MeV for up and down quarks (see the discussion in Ref. [6]). In either case, the Green function (3) at the stationary field is diagonal in flavor space.

Extremizing (10) with respect to $\Sigma_0, \Psi_a, \bar{\Psi}_a$ yields the field equations

$$iG_{\Sigma_0}^{-1} \Psi_a(x) = \left[i\partial\!\!\!/ - \Sigma_0(x) \right] \Psi_a(x) = 0, \tag{12}$$

$$\frac{1}{g_0} \Sigma_0(x) = \mathrm{tr}_\gamma \mathrm{tr}_f G_{\Sigma_0}(x,x) - \frac{1}{N_c} \bar{\Psi}_a(x) \Psi_a(x). \tag{13}$$

In the absence of external quark sources, the ground state expectation value of a fermion field is always zero and the expectation value $\Sigma_0(x)$ is constant, so that (10) reduces into

$$\frac{\Gamma(\Sigma_0)}{N_c} = -i\mathrm{Tr}' \ln iG_{\Sigma_0}^{-1} - \frac{1}{2g_0} \int d^D x {\Sigma_0}^2. \tag{14}$$

Extremizing this with respect to Σ_0, we find the gap equation

$$\frac{1}{g_0} = i(\mathrm{tr}_f \mathbb{1})(\mathrm{tr}_\gamma \mathbb{1}) \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 - \Sigma_0^2} \tag{15}$$

which becomes, after a Wick rotation to euclidean momenta p_E with $p_0 = ip_{E,0}$, $id^Dp \to -d^Dp_E$, $p^2 \to -p_E^2$,

$$\frac{1}{g_0} = 2 \times 2^{D/2} \int \frac{d^D p_E}{(2\pi)^D} \frac{1}{p_E^2 + \Sigma_0^2}. \tag{16}$$

We have devided the two sides of the gap equation by a common factor Σ_0, supposing that we are in the spontaneously broken phase.

This equation must be regularized, which may be done in many ways. Here, we shall use two methods: the analytic continuation in the dimension D, and a cutoff Λ in momentum space. The former is mathematically more elegant and has the advantage of allowing us to relate the properties in four dimensions with those in $2+\varepsilon$. The second is more physical since it shows us the true divergence caused by the infinite number of degrees of freedom of the field system. For rotationally symmetric integrands, we may factorize the integration measure in momemtum space as follows:

$$\int_{-\infty}^{\infty} \frac{d^D p_E}{(2\pi)^D} = \frac{2}{(2\pi)^D} \frac{\pi^{D/2}}{\Gamma(D/2)} \int_0^{\infty} dp_E\, p_E^{D-1} = \frac{1}{(2\pi)^D} \frac{\pi^{D/2}}{\Gamma(D/2)} \int_0^{\infty} dp_E^2\, (p_E^2)^{D/2-1}, \tag{17}$$

bringing the gap equation (16) to the form

$$\frac{1}{g_0} = 2 \times 2^{D/2} \frac{1}{(2\pi)^D} \frac{\pi^{D/2}}{\Gamma(D/2)} \int dp_E^2 (p_E^2)^{D/2-1} \frac{1}{p_E^2 + \Sigma_0^2} \tag{18}$$

Using the integral formula $\int_0^{\infty} dx x^{\alpha-1}(1+x)^{-\alpha-\beta} = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$, we arrive at

$$\frac{1}{g_0} = 2\Sigma_0^{D-2} \frac{\Gamma(1-D/2)}{(2\pi)^{D/2}}. \tag{19}$$

The volume density of the effective action (14), the *effective potential* per quark, is

$$v(\Sigma_0) \equiv -\frac{\Gamma(\Sigma_0)}{\Omega N_c} = \frac{1}{2g_0}{\Sigma_0}^2 + i \mathrm{tr}_\gamma \mathrm{tr}_f \int \frac{d^D p}{(2\pi)^D} \ln\left(\not{p} - \Sigma_0\right). \tag{20}$$

where Ω denotes the D-dimensional volume $\int d^D x$.

Performing the internal traces, we obtain in euclidean space

$$v(\Sigma_0) = \frac{1}{2g_0}{\Sigma_0}^2 - 2 \times 2^{D/2} \int \frac{d^D p_E}{(2\pi)^D} \frac{1}{2} \ln\left(p_E^2 + {\Sigma_0}^2\right), \tag{21}$$

which becomes, with (17),

$$v(\Sigma_0) = \frac{1}{2g_0}{\Sigma_0}^2 - \frac{1}{(2\pi)^{D/2}\Gamma(D/2)} \int dp_E^2 (p_E^2)^{D/2-1} \ln\left(p_E^2 + {\Sigma_0}^2\right). \tag{22}$$

This expression contains a divergent constant term associated with the chirally symmetric state with $\Sigma_0 = 0$. Subtracting this gives an energy difference, the *condensation energy*:

$$\begin{aligned} \Delta v(\Sigma_0) &= \frac{1}{2g_0}{\Sigma_0}^2 - \frac{1}{(2\pi)^{D/2}\Gamma(D/2)} \int dx x^{D/2-1} \ln\left(1 + \frac{{\Sigma_0}^2}{x}\right) \\ &= \frac{1}{2}\left[\frac{1}{g_0}{\Sigma_0}^2 - {\Sigma_0}^D \frac{4}{D} \frac{1}{(2\pi)^{D/2}} \Gamma(1-D/2)\right]. \end{aligned} \tag{23}$$

For even dimensions D, both the gap equation (19) and the effective potential (23) are divergent due to a pole in the factor $\Gamma(1-D/2)$. Introducing the diverging parameter

$$b_\epsilon = \frac{2}{D}\frac{\Gamma(1-D/2)}{(2\pi)^{D/2}}, \tag{24}$$

we can rewrite the gap equation and effective potential in the more compact form as

$$\frac{1}{g_0} = D\Sigma_0^{D-2} b_\epsilon, \tag{25}$$

$$\Delta v(\Sigma_0) = \frac{1}{2}\left[\frac{1}{g_0}{\Sigma_0}^2 - 2{\Sigma_0}^D b_\epsilon\right]. \tag{26}$$

In the more physical form involving a cutoff Λ in momentum space to regularize all integrals, these expressions look more complicated:

$$\frac{1}{g_0} = \frac{2}{(2\pi)^2}\left[\Lambda^2 - \Sigma_0^2 \ln\left(1+\frac{\Lambda^2}{\Sigma_0^2}\right)\right], \tag{27}$$

$$\Delta v(\Sigma_0) = \frac{1}{2}\left\{\frac{1}{g_0}{\Sigma_0}^2 - \frac{2}{(2\pi)^2}\left[\frac{{\Sigma_0}^2\Lambda^2}{2} + \frac{\Lambda^4}{2}\ln\left(1+\frac{{\Sigma_0}^2}{\Lambda^2}\right) - \frac{{\Sigma_0}^4}{2}\ln\left(1+\frac{\Lambda^2}{{\Sigma_0}^2}\right)\right]\right\}. \tag{28}$$

IV. CHIRAL FLUCTUATIONS

Since the physical number of quarks N_c is finite, the fields perform fluctuations of the magnitude $\approx 1/\sqrt{N_c}$ around their extremal value. As long as N_c can be considered as a large number, the deviation from the extremal field configuration

$$(\sigma', \pi'_a) \equiv (\sigma - \Sigma_0, \pi_a) \tag{29}$$

are small, and the action can be expanded in powers of (σ', π'_a). The quadratic terms in this expansion define the propagators of the collective fields (σ', π'_a). The higher expansion terms of the trace of the logarithm in (8) define the interactions. With the decomposition (29), the propagator (3) of the quarks may be decomposed into a constant part and fluctuations as

$$iG^{-1} = iG_{\Sigma_0}^{-1} - (\sigma' + i\gamma_5\lambda_a\pi'_a), \tag{30}$$

with G_{Σ_0} of Eq. (11). Since $\Sigma_0 \neq 0$ is equal to the constituent quark mass M, we shall from now on replace it by M everywhere. Then we have

$$\mathrm{Tr}\ln iG^{-1} = \mathrm{Tr}\ln iG_M^{-1} + \mathrm{Tr}\ln\left[1 + iG_M\left(\sigma' + i\gamma_5\lambda_a\pi'_a\right)\right]. \tag{31}$$

An expansion up to the second order gives

$$\mathrm{Tr}\ln iG^{-1} \approx \mathrm{Tr}\ln iG_M^{-1} + \mathrm{Tr}\left[iG_M\left(\sigma' + i\gamma_5\lambda_a\pi'_a\right)\right] - \frac{1}{2}\mathrm{Tr}\left[iG_M\left(\sigma' + i\gamma_5\lambda_a\pi'_a\right)\right]^2, \tag{32}$$

leading to an approximate partition function

$$\mathcal{Z} = \mathcal{Z}_0 \int \mathcal{D}\sigma\mathcal{D}\pi \exp\left(iN_c\left\{\frac{i}{2}\mathrm{Tr}'\left[iG_M\left(\sigma' + i\gamma_5\lambda_a\pi'_a\right)\right]^2 - \frac{1}{2g_0}\int d^Dx({\sigma'}^2 + {\pi'_a}^2)\right\}\right). \tag{33}$$

where

$$\mathcal{Z}_0 \equiv \exp\left[-\Omega_E N_c v(M)\right], \tag{34}$$

with Ω_E being the euclidean volume. The inverse of the expression between the fields in the exponent of Eq. (33) gives us directly the desired collective free field propagators $G_{\sigma'}, G_{\pi'}$. In momentum space we have

$$\mathcal{A}_0[\sigma',\pi'] = \frac{1}{2}\int d^D q\left[\pi'_a(q)G_\pi^{-1}\pi'_a(-q) + \sigma'(q)G_\sigma^{-1}\sigma'(-q)\right], \tag{35}$$

where

$$G_\sigma^{-1} = N_c\left\{2\times 2^{D/2}\int\frac{d^D p_E}{(2\pi)^D}\frac{(p_E^2 + p_E q_E - M^2)}{(p_E^2+M^2)[(p_E+q_E)^2+M^2]} - \frac{1}{g_0}\right\}, \tag{36}$$

$$G_\pi^{-1} = N_c\left\{2\times 2^{D/2}\int\frac{d^D p_E}{(2\pi)^D}\frac{(p_E^2 + p_E q_E + M^2)}{(p_E^2+M^2)[(p_E+q_E)^2+M^2]} - \frac{1}{g_0}\right\}. \tag{37}$$

Combining (36) and (37) with the gap equation (16), we eliminate the term $1/g_0$ and obtain for the two propagators

$$G_{\sigma,\pi}^{-1} = 2\times 2^{D/2}N_c\int\frac{d^D p_E}{(2\pi)^D}\frac{1}{(p_E^2+M^2)[(p_E+q_E)^2+M^2]}\left[-q_E^2 - p_E q_E - (2M^2,0)\right]. \tag{38}$$

The notation $(2M^2,0)$ indicates that only the equation for σ contains an extra term $2M^2$. Performing the momentum integrals with dimensional regularization we obtain, with the help of the formula $(ab)^{-1} = \int_0^1 dy\,[(a-b)y+b]^{-2}$, the inverse propagators

$$G_{\sigma,\pi}^{-1} = 2\times 2^{D/2}N_c\int_0^1 dy\int\frac{d^D p_E}{(2\pi)^D}\frac{1}{[(q_E^2+2p_E q_E)y + p_E^2 + M^2]^2}\left[-q_E^2 - p_E q_E - (2M^2,0)\right]. \tag{39}$$

After a shift of integration variables $p_E \to p_E - q_E y$, we arrive at

$$G_{\sigma,\pi}^{-1} = -2N_c\frac{\Gamma(2-D/2)}{(2\pi)^{D/2}}\int_0^1 dy\frac{q_E^2(1-y) + (2M^2,0)}{[M^2 + q_E^2 y(1-y)]^{2-D/2}}. \tag{40}$$

In four spacetime dimensions, the denominator is unity, and the integral reduces to $q_E^2/2$ for the pseudoscalars, and $(q_E^2+4M^2)/2$ for the scalars. The first leads to a zero mass for pions as a manifestation of Goldstone's theorem, the second a mass equal to two constituent quark masses for the σ-mesons. To have a finite result, we must regularize these expressions. Going to $D = 4-\epsilon$ dimensions, the inverse euclidean propagator is seen to start out for small q_E^2 like

$$G_\pi^{-1} = -2N_c\frac{\Gamma(2-D/2)}{(2\pi)^{D/2}M^{4-D}}\frac{q_E^2}{2} + \mathcal{O}(q_E^4). \tag{41}$$

Inserting b_ϵ from Eq. (24), this becomes

$$G_\pi^{-1} \approx -N_c\left(1-\frac{D}{2}\right)Db_\epsilon\frac{1}{M^{4-D}}\frac{q_E^2}{2} \equiv -Z_\pi^{(\epsilon)}q_E^2. \tag{42}$$

If the theory is regularized with a cutoff Λ in momentum space, this becomes

$$G_\pi^{-1} = -\frac{N_c}{(2\pi)^2}\left[\ln\left(1+\frac{\Lambda^2}{M^2}\right) - \frac{\Lambda^2}{\Lambda^2+M^2}\right]q_E^2 \equiv -Z_\pi^{(\Lambda)}q_E^2. \tag{43}$$

In the last two equations, the factors in front of q_E^2 have been identified as the wave function renormalization constants Z_π of the pion field in the two regularization schemes.

As a consequence of the spontaneous symmetry breakdown, the fluctuations of the pseudoscalar fields are massless. These fields appear in the x-space version of the action (35) in a pure gradient form. Going over to unit vector fields $(\hat\sigma',\hat\pi'_a) \equiv (\sigma',\pi'_a)/M$, we rewrite this as

$$\mathcal{A}_0[\pi'] = \frac{\beta}{2}\int d^D x\left\{[\partial\hat\pi'_a(x)]^2\right\}. \tag{44}$$

In this normalization, the prefactor β is called the *stiffness* of the field fluctuations [5,7–9]. Our result (42) shows that the stiffness of pion fluctuations in the Nambu–Jona-Lasinio model is given by

$$\beta = M^2 Z_\pi^{(\epsilon)}. \tag{45}$$

Note that for D=2, the stiffness becomes

$$\beta = \frac{N_c}{2\pi}, \tag{46}$$

and coincides with the stiffness calculated in Ref. [5] for the Gross-Neveu model (which contains a factor N which has to be identified with $N_f \times N_c = 2N_c$).

With a cutoff regularization in $D = 4$ dimensions, the stiffness of $\hat{\pi}'_a$-fluctuations is

$$\beta = M^2 Z_\pi^{(\Lambda)} = \frac{N_c}{(2\pi)^2} M^2 \left\{ \ln\left[1 + \left(\frac{\Lambda}{M}\right)^2\right] - \frac{\Lambda^2}{M^2 + \Lambda^2} \right\}. \tag{47}$$

This is the crucial quantity on which our fatal conclusions for the lack of spontaneous symmetry breaking will be based. We shall now demonstrate that the stiffness (47) is far too small to let the spontaneous symmetry breakdown occur. The disordering effect of phase fluctuations is well-knwon from many model studies of the 0(4)-symmetric Heisenberg model on a lattice. High-temperature expansions and Monte Carlo simulations show that there exists a critical stiffness below which the system goes over into a disordered state. For an analytic estimate of this critical stiffness we use an 0(4)-symmetric nonlinear sigma model whose Lagrangian is parametrized in term of the same stiffness as in (44)

$$\mathcal{L} = \frac{\beta}{2} (\partial n_i)^2, \qquad i = 1, \ldots, N_n, \tag{48}$$

where n_i is a fluctuating unit vector with $N_n = 4$ components which we identify with the normalized field vector $(\hat{\sigma}', \hat{\pi}'_a)$. In order to calculate the critical stiffness, we relax the unit vector constraint by introducing an additional field $\lambda(x)$ playing the role of a Lagrange multiplier. The n_i-fields can be integrated out in the partition function, and leads to an action

$$S = \frac{N_n}{2} \mathrm{Tr} \ln \left[-\partial^2 + \lambda(x)\right] - \beta \int d^D x \frac{\lambda(x)}{2}, \tag{49}$$

where Tr denotes the functional trace (the summation over the fields component has already been performed). For a large number N_n of components, the fluctuations are suppressed and the field $\lambda(x)$ becomes a constant, satisfying the gap equation

$$\beta = N_n \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + \lambda}. \tag{50}$$

If there is a nonzero solution $\lambda \neq 0$, this will play the role of a mass of the n_i-fluctuations. The model has a phase transition at a critical stiffness

$$\beta_c = N_n \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2}. \tag{51}$$

For a smaller stiffness, the phase fluctuations are so violent that the system goes into a disordered phase with $\lambda \neq 0$ giving all fields n_i a nonzero mass λ. Since the fields n_i are the normalized σ and π_a fields of the model, this implies an equal nonzero mass of σ and π_a mesons, and thus a restoration of chiral symmetry. In our model, the number N_n is equal to four, which is not very large. Fortunately, Monte-Carlo studies of the model have shown that $N_n = 4$ is large enough to ensure the existence of the transition, and that the critical stiffness obtained from (51) is correct to within 20%.

For $N_n = 4$ and a cutoff Λ, the critical stiffness is given by

$$\beta_c = 4\frac{\Lambda^2}{16\pi^2}. \tag{52}$$

By comparing (52) with the stiffness of the model in (47), we find the equation

$$N_c = \left(\frac{\Lambda}{M}\right)^2 \left\{\ln\left[1+\left(\frac{\Lambda}{M}\right)^2\right] - \frac{(\Lambda/M)^2}{1+(\Lambda/M)^2}\right\}^{-1}. \tag{53}$$

This equation determines the number N_c of identical quarks which is necessary to produce a sufficient stiffness β to prevent the restoration of chiral symmetry. Only if the number of colors exceeds this critical value of N_c, would the model really possess a phase in which the pion is a Goldstone boson and the σ-meson a massive meson whose mass is twice as large as that of the constituent quarks. The critical number (53) is plotted in Fig. 1. We see that $N_c = 5$ would be the smallest allowed value. This number, however, is incompatible with color SU(3). This suggests strongly that the Nambu–Jona-Lasinio model remains always in the symmetric phase due to chiral fluctuations, and cannot be used to describe the chiral symmetry breakdown in quark physics, as has been claimed by many publications, in particular in nuclear physics [3].

It is interesting to see that the same conclusion cannot be reached in the dimensional regularizaion scheme. There the integral in (51) determinig the critical stiffness vanishes. This is one of the typical unphysical features of dimensional regularization [10], showing that this scheme can meaningfully be used only in renormalizable theories, where all quantities which diverge in a cutoff regularization are physically irrelevant. Here they are not!

For completeness, let us estimate the size of the stiffness parameter β numerically. It can be obtained from the decay constant f_π which is oservable in the leptonic decay of charged pions. It is defined by the matrix element of the axial vector current $A_j^\mu = \bar\psi\gamma_\mu\gamma_5(\lambda_j/2)\psi$ between a pion state at rest and the vacuum:

$$< 0|\partial_\mu A_k^\mu|\pi_i(p) >= f_\pi m_\pi^2 \delta_{ki}. \tag{54}$$

In order to caluclate this matrix element within the Nambu–Jona-Lasinio model, we have to allow for a small mass of the pions by keeping a small nonzero bare quark mass m_0 in the Lagrangian (1). This gives the pions a small mass equal to

$$m_\pi^2 = \frac{N_c m_0}{g_0 Z_\pi M}. \tag{55}$$

On the other hand, the divergence of the axial current in (54) can be evaluated using the equation of motion of the pseudoscalar fields, being

$$\partial_\mu A_k^\mu = -\frac{i}{2}\bar\psi\gamma_5\{m_0,\lambda_k\}\psi = m_0\frac{N_c}{g_0}\pi_k. \tag{56}$$

The right hand side contains the pion field π_k of the model whose kinetic term contains the prefactor Z_π. When taking the matrix element of (56) between a pion state and the vacuum, both of unit normalization, we obtain a factor $Z_\pi^{-1/2}$. Thus we find

$$f_\pi^2 = M^2 Z_\pi = \beta. \tag{57}$$

This equality is independent of the regularization scheme.

The experimental value for f_π is

$$f_\pi = 0.093 \text{ GeV} \tag{58}$$

so that the stiffness of the chiral fluctuations is

$$\beta = 0.00865 \text{ GeV}^2. \tag{59}$$

In order to estimate the critical stiffness, we must specify the cutoff to be used in this model. Inserting (59) into (47) and inserting for the constituent mass M half the σ-meson mass, which we take as $m_\sigma \approx 0.6$ GeV, we find that for $N_c = 1$, the cutoff must be

$$\Lambda = 3.27 \text{ GeV} \tag{60}$$

to adapt the model to the σ - π system in nature. For $N_c = 3$, we would find the smaller value $\Lambda = 0.825$ GeV. Inserting this into Eq. (52) for the critical stiffness, we find $\beta_c \approx 0.271 \text{ GeV}^2$ for $N_c = 1$ and $\beta_c \approx 0.0172 \text{ GeV}^2$ for $N_c = 3$. Both values are much larger than the model stiffness in (59), such that the model always remains in the disordered phase. The broken symmetry is restored by chiral fluctuations and σ and π exist with equal masses $m_\sigma^2 = m_\pi^2 = \lambda$.

Our conclusions were derived from a study of only the σ, π fields. The inclusion of other flavors does not help preventing the restoration since the associated pseudoscalar mesons are quite massive, making their fluctuations irrelevant to the described phenomenon.

V. CONCLUSION

We have shown that within a certain nonperturbative approximation the Nambu–Jona-Lasinio model does not really display the spontanous symmetry breakdown for whose illustration it was contructed. The fluctuations of σ- and π_a-fields restore chiral symmetry and make σ and π equally massive. If our conclusion survives in higher approximations, this would invalidate a large number of publications, especially in nuclear physics, which have been based on the existence of a symmetry-broken ground state of the model. In particular, all studies of the temperature dependence of the symmetry-broken state [3], and of the classical solution [11] of the collective field Eq. (9), would deal with nonexisting objects, thus calling for further investigations.

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[10] Such unphysical features occur in many loop calculations. For instance, the effective potential (26) is unphysical in $D = 3$, since b_ϵ is finite in odd dimensions. The cutoff version of the integral over the logarithm in (22) diverges with the cutoff like

$$\int d^3p \ln(1 + m^2/p^2) \propto \left\{\Lambda^3 \ln[1 + (m/\Lambda)^2] + 2m^2[\Lambda - m \arctan(\Lambda/m)]\right\} \sim 3m^2\Lambda - \pi m^3 + \mathcal{O}(1/\Lambda). \quad (61)$$

The dimensionlly regulized integral yields only the finite value $-\pi m^3$ and misses out on the large positive energy $3m^2\Lambda$, which is physical in an unrenormalizable theory.

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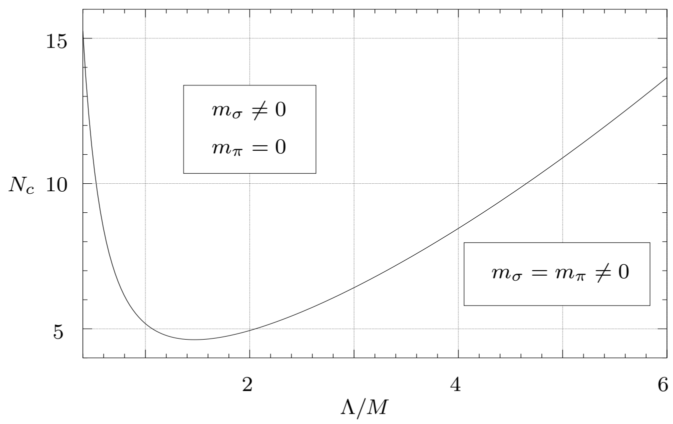

FIG. 1. N_c, extracted from condition Eq. (53), as a function of Λ/M